Literature Review: Quantum Control

The maximum possible control time for a quantum computer is a crucial parameter because this describes how much time is available to perform some computation before the system relaxes and noise affects the results. The goal for this project is to find a relationship between time taken for a given computation and the depth for the respective commutator in the Lie algebra.

Quantum Computing

Quantum computers utilize quantum phenomena to perform operations which are not possible on a conventional computer. Information is stored on qubits in the place of bits, these are represented as two-state mechanical systems – usually the spin state of particles. This means qubits are able to store a superposition of states simultaneously. However one of the main limits to development in quantum computing is decoherence. This is caused by many factors such as interaction with the environment and implicit system noise.

As is discussed in [1] this noise is proportional to the time taken for a computation. The authors of this paper determined that it was infeasible to achieve negligible noise due to faster control fields as the methods involved would exacerbate the presence of noise. It is therefore important to explore how to optimize the control time of a computation without increasing the amplitude of the control field. This would have many implications in the field of quantum computing. These include the ability to solve np-complete problems and integer factorization on the product of large primes, allowing for advances in areas such as cryptography.

Lie Algebras

The initial text recommended for reading was *Introduction to Quantum Control and Dynamics* [2]*;* this provided a firm ground level from which an understanding of the theory behind quantum control could be developed. The basic Dirac notation is introduced and these can describe the states of a system such as the initial and final goal state. It then shows that a unitary transformation operator (a matrix) can act on an initial state $\overrightarrow{\psi_o}$ such that:

$$\overrightarrow{\psi_t} = U(t)\overrightarrow{\psi_o}$$

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Where $U(t)$ is a solution to the Schrödinger equation,

$$i\dot{U}(t) = H(u(t))U(t)$$

$H(u(t))$ is the control Hamiltonian of the system and $U(t)$ is an element of the Lie group of unitary matrices. From this we can see that the computation required to perform this transfer between states is given by $U(t)$.

The text then introduces the Lie algebras, these are a set of matrices which satisfy the following statements:

$$X, Y \in L \qquad \alpha X + \beta Y \in L \quad \text{If } \alpha, \beta \in R \qquad [X, Y] \in L$$

The system of interest we are investigating consists of a long chain of spin states located at sites on the chain.

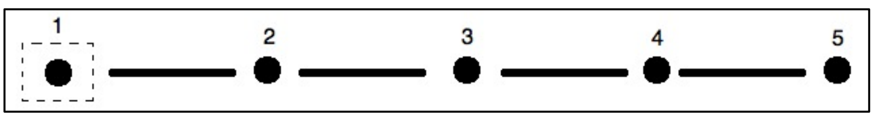

Figure 1

Two Hamiltonians, the system Hamiltonian H_0 and the control Hamiltonian H_C govern this system; H_C corresponds to a magnetic field acting on site 1. By creating a Lie algebra from these two Hamiltonians you can understand what is possible to control within the system.

The Lie Tree

To develop a Lie algebra it is sufficient to produce a simple Lie tree, this uses two initial matrices contained within the Lie algebra, H_0 and H_C, and the commutator between the pair to create the top of the tree, which can be seen in figure 2.

The commutator of two matrices is defined as:

$$[X, Y] = XY - YX$$

The properties of which are:

$$[X, X] = 0 \qquad [X, Y] = -[Y, X]$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

The Jacobi Identity

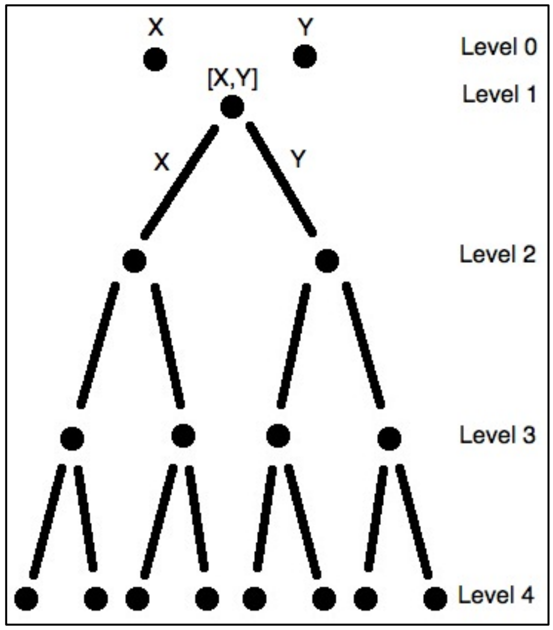

Figure 2

This tree is formed by taking the commutator of each point of the tree with either the X or Y Hamiltonian, representing either left or right on the tree. If a point on the tree may be expressed through linear combinations of previous elements of the Lie tree then it is not a new element of the Lie algebra. Eventually by following the tree down, there will be no new additions to the algebra and this forms the basis of the dynamical Lie algebra.

The Lie tree can be shown to be enough to describe the entire Lie basis through use of the Jacobi identity. If some point on the tree were to be commutated with the previous point then it can be shown that an identical matrix can be created just by commutating the previous point with the initial matrices.

$$[[A,[A,B]],[A,B]] = -[A,[B,[A,[A,B]]]] - [B,[[A,[A,B]],A]]$$

In his book Bilinear Control systems [3], David Elliot demonstrates an algorithm in Mathematica which calculates the Lie Tree for a given system as is shown in figure 3.

```
Rank::Usage="Rank[M]; matrix M(x); output is generic rank."
Rank[M_]:= Length[M[[1]]]-Dimensions[NullSpace[M]][[1]];
Dim::Usage="Dim[L]; L is a list of n by n matrices. \n
Result: dimension of span(L). "
Dim[L_]:=Rank[Table[Flatten[L[[i]]],{i,Length[L]}]];

LieTree::Usage="LieTree[gen,n]; \n
gen=list of n x n matrices; output=basis of \n
the Lie algebra generated by gen."
LieTree[gen_, n_]:= Module[{db,G, i,j,k, newLevel,newdb,lt,
L={},nu=n^2,tree,T,m=Length[gen],r=Dim[gen]},
Basis=gen; If[r<m,Return["Dependent list"]];
LieWord=""; T=Array[tree, nu]; tree[1]=gen;
For[i=2, i<nu-1, i++,{L=tree[i-1];Lm=Length[L];
db=Dim[Basis]; newLevel={};
For[j=1,j<m+1,j++,{For[k=1,k<Lm+1,k++,
{G=gen[[k]];com=G.L[[j]]-L[[j]].G;
trial= Append[Basis, com]; newdb= Dim[trial];
If[newdb >db, {newLevel=Append[newLevel,com];
Basis=Append[Basis, com];
LieWord=StringJoin[LieWord,ToString[{i,j,k}]];
db= newdb;};];};];};];
tree[i]=newLevel; lt=Length[newLevel];
If[lt <1, Break[], r =lt]};]; Basis];
```

Figure 3

This algorithm accepts a matrix input and calculates the Lie algebra associated with it. There also exists a Rank function which can be used to determine if a commutator is novel within the Lie Tree. It is the goal of this project to produce a similar algorithm in python which automatically calculates the Lie Algebra for a system and discards commutators which are linearly dependent on previous results. This could then be used to determine the control time at different depths.

Controllability

Once the complete Lie algebra has been created for a given system then you can find a matrix relating to the control of a particular quantum state on this chain. Therefore you are *indirectly* controlling the entire chain of spin states through the *direct* control of the spin of the system at site 1 in figure 1. Work by D'Alessandro further illustrates this idea of controllability when limited to just two quantum states and states that indirect controllability of a state is equivalent to *complete* controllability. [4] [5]

The degree of controllability is discussed in [6] and [7], they describe *operator controllable* as when there exists a unitary operation which can transfer the state of the system from the identity to some final state $\overrightarrow{\psi_t}$. They also explain density-matrix, pure-state, equivalent-state and observable control; these all offer different methods to achieve the best control of a particular system, but this system assumes operator controllability.

The Magnus Expansion

In order to determine how depth affects the control time for a given computation it is necessary to describe our computation in terms of a time-dependent Hamiltonian:

$$H(t) = H_0 + f(t)H_C$$

Where $f(t)$ is simply a function which changes over time. Using the Schrodinger equation, this expression can be expressed with the time evolution operator $U(t)$ so that:

$$\Omega(t) = e^{-\frac{Ht}{\hbar}}$$

$$U(t) = e^{\Omega(t)}$$

It is then possible to use the Magnus expansion [8] to calculate $\Omega(t)$:

$$\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t)$$

For which the first three terms are:

$$\Omega_1(t) = \int_0^t H(t_1)\,dt_1$$

$$\Omega_2(t) = \frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 [H(t_1), H(t_2)]$$

$$\Omega_3(t) = \frac{1}{6}\int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3 ([H(t_1),[H(t_2),H(t_3)]] + [H(t_3),[H(t_2),H(t_1)]])$$

It is of course essential that something approximating the time optimal control is used for any comparison, as it is likely that there will be more than one evolution which yields the same result.

Time Optimization

There are many potential methods of determining time optimal controls discussed in depth in [9]. In this paper one method is put forward to calculate the minimal time needed to move from an initial state to a desired state, utilizing manipulations of the Bloch sphere.

However this does not give the control necessary – only a time which can be compared to a given control time to determine if it is optimal. It is suggested that this is done iteratively using a closed-loop learning control. This involves a learning algorithm which adapts the control repeatedly until the time approximates the minimum.

Work done by Khaneja [10] shows that the minimum time required to produce F can be expressed as the shortest path in the space G/K, the drifting process; where G is the compact semi-simple Lie group and K is a compact subgroup of G. It also shows that the transfer efficiency $\eta(t)$ from a starting operator $\rho(0)$ to a final operator F is given by:

$$\eta(t) = \left\| tr(F^{\dagger} U(t) \rho(0) U^{\dagger}(t)) \right\|$$

This has a maximum value of 1, relating to a complete transfer of some operation on the system before it becomes decoherent. In an attempt to optimize the control of a system using a singe control pulse [11] to achieve complete controllability; Schirmer found that in most anharmonic, non-decomposable systems, complete control was achievable as well as for systems with equally spaced energy levels given the correct initial dipole moment conditions. However for a standard harmonic oscillator, the system was not completely controllable because there were unitary operators that were not reachable.

In [12] it is also suggested that a feedback mechanism is the most feasible method for optimizing control times. However, the author suggests minimizing a Hamiltonian containing the optimal control and trajectory. This has the advantage that it can be utilized for single- or multi-input systems, however does require the optimal trajectory and control.

Conclusion

After reviewing the literature we conclude that it is important to reduce control times where possible to avoid introducing noise. However this is inhibited as increasing the control field will cause additional noise meaning it is important to explore alternative methods. As such this project intends to explore the relationship between the depth of commutator necessary for a computation with the time taken. The Magnus expansion appears to provide the best way to determine the control time of a computation. When combined with a feedback system it is possible to ensure time-optimal control of a system. The final goal is to design an algorithm in Python capable of calculating the Lie algebra of a system, given H_0 and H_C, and comparing the control times of each level of the Lie Tree. The potential applications of this extend into areas such as solving np-complete problems and quantum cryptography.

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